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EE 301 – Difference Equations

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These notes present some information about difference equations supplementing the material given in the lectures. The goal is to present some important aspects of the difference equations which is not thoroughly presented in the textbook.

1. Difference Equations

Difference equations are the discrete-time duals of the differential equations. Our main interest is the discussion of the Constant Coefficient Difference Equations (CCDE). This class of difference equations closely resembles the corresponding class in the continuous time.

A discrete-time LTI system with the input $x[n]$ and output $y[n]$ can be expressed as follows:

$$y[n]+\sum_{k=1}^{M}\alpha_k y[n-k]=\sum_{k=0}^{L}\beta_k x[n-k]. \qquad (1)$$

It should be noted from $y[n]=-\sum_{k=1}^{M}\alpha_k y[n-k]+\sum_{k=0}^{L}\beta_k x[n-k]$ that $y[n]$ is simply a linear combination of the input samples $\{x[n],x[n-1],...,x[n-L]\}$ and the output samples $\{y[n-1],y[n-2],...,y[n-M]\}$. Hence, every output sample $y[n]$ can be easily calculated if we have the input sequence $x[n]$ (which is the external input to the system) and prior output samples, that is $y[n-k]$ for $k=\{1,2,...,M\}$.

We temporarily call the right hand side (RHS) of (1) as $f[n]$, that is $f[n]=\sum_{k=0}^{L}\beta_k x[n-k]$. The actual input to the system is $x[n]$; but for the calculation of the output sample ($y[n]$), $f[n]=\sum_{k=0}^{L}\beta_k x[n-k]$ is required. Since $x[n]$ is the external input provided to us, we may equivalently consider that $f[n]=\sum_{k=0}^{L}\beta_k x[n-k]$ is the effective input to the system and can write the following difference equation for system description:

$$y[n]+\sum_{k=1}^{M}\alpha_k y[n-k]=f[n]. \qquad (2)$$

The mentioned decomposition is also illustrated in Figure 1. Iterating one more time, $x[n]$ is an arbitrary input which is beyond our control and $f[n]$ is the secondary input derived from $x[n]$ and processing $f[n]$ through (2) yields y[n].

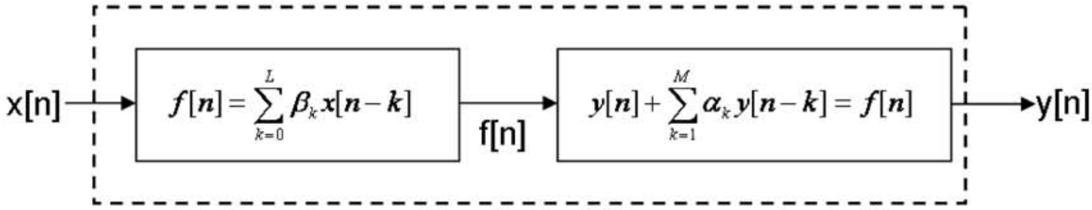

Fig.1 : A decomposition for the difference equation in (1)

From Figure 1, it can also be noted that the relation between $f[n]$ and $x[n]$ is convolution type, that is $f[n]=g[n]*x[n]$ where $g[n]=\sum_{k=0}^{L}\beta_k\delta[n-k]$ is the impulse response of the system generating $f[n]$ from $x[n]$.

Without any loss generality, we work on the difference equation $y[n]+\sum_{k=1}^{M}\alpha_k y[n-k]=f[n]$ in the remaining part of these notes. Our goal is to outline the solution of CCDE difference equations through the study of this form.

2. Homogeneous Solution

We present the topic of homogenous solution with an example. The example is on the generation of n'th Fibonacci number.

Fibonacci sequence is defined as $[1,1,2,3,5,8,13,...]$. This sequence comes up frequently in many problems and attracted the attention of public through some popular culture items (http://en.wikipedia.org/wiki/Fibonacci_numbers_in_popular_culture). Each term in this sequence is the sum of two terms prior to that term. For example, the term of 5 in $[1,1,2,3,5,8,13,...]$ is the sum of 2 and 3; the term 8 is sum of 5 and 3 etc. More succinctly, if we call the n'th term of the sequence as $y[n]$, then the recursion of

$$y[n]=y[n-1]+y[n-2]$$

generates the sequence of Fibonacci numbers.

To initiate the recursion, we need initial values for $y[n]$ sequence. It should be clear that two initial values are required to start this recursion. (More generally, for a difference equation as in (2), we need M initial conditions.) The goal becomes calculation of the n'th term of the series given the initial conditions, that is the solution of the following difference equation:

$$\begin{aligned} &y[n]=y[n-1]+y[n-2] \\ &y[0]=1 \\ &y[1]=1 \end{aligned} \quad . \tag{3}$$

It should be noted that we have arbitrarily selected to the starting index for the Fibonacci from 0. Hence, the first Fibonacci number corresponds to $y[0]$ and therefore, two initial conditions generating the Fibonacci sequence are $y[0]=y[1]=1$ in (3). We could have selected any other index such as $y[-1]=y[0]=1$ or $y[5]=y[6]=1$. The other choices do not change the value of the Fibonacci numbers; how changes the way you refer to them, i.e. changes only their index! Figure 2 shows the effect of two different indices.

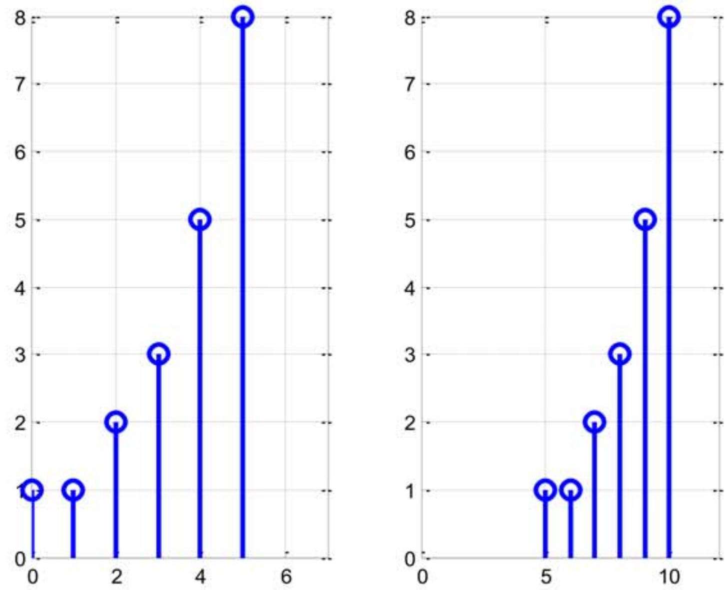

Fig.2 : First 6 elements of the Fibonacci sequence with two different indexing.

The solution of homogenous CCDE equations is accomplished by guessing the structure of the solution (as in its differential equations counterpart). For the homogenous solution, we have the external input, that is the forcing term, $f[n]=0$. This leads to the following difference equation:

$$y_h[n]+\sum_{k=1}^{M}\alpha_k y_h[n-k]=0\,. \tag{4}$$

In the equation above, $y_h[n]$ refers to the homogenous part of the solution. To find the homogenous solution, we make the guess of $y_h[n]=Ar^n$. It should be noted that when $A=0$ or $r=0$, $y_h[n]$ reduces to the zero sequence, $y_h[n]=0$. It should be clear that $y_h[n]=0$ satisfies the difference equations; but having $y_h[n]=0$ for all n values is clearly not an interesting solution and called the trivial solution. In general, the trivial solution is not the solution that we are looking for, since this solution does not satisfy the initial conditions. Our goal is to find the non-trivial solutions of the difference equations so that we can generate results of interest such as the ones shown in Figure 2.

By substituting $y_h[n]=Ar^n$ into $y_h[n]+\sum_{k=1}^{M}\alpha_k y_h[n-k]=0$, we get the relation $A(r^n+\sum_{k=1}^{M}\alpha_k r^{n-k})=0$. The same equation can be written as $Ar^{n-M}(r^M+\sum_{k=1}^{M}\alpha_k r^{M-k})=0$. It should be remember that the recursion holds for all n values. Hence, we need to find A and r values ($y_h[n]=Ar^n$) such that $Ar^{n-M}(r^M+\sum_{k=1}^{M}\alpha_k r^{M-k})=0$ is satisfied for all n.

Since we are looking for non-trivial solution, we can divide both sides of $Ar^{n-M}(r^M + \sum_{k=1}^{M}\alpha_k r^{M-k}) = 0$ by Ar^{n-M} and get the following equation:

$$\text{Characteristic Equation:} \quad r^M + \sum_{k=1}^{M}\alpha_k r^{M-k} = 0$$

The roots of this equations satisfy the equation $Ar^{n-M}(r^M + \sum_{k=1}^{M}\alpha_k r^{M-k}) = 0$, hence the recursion for all n values. For an M'th degree polynomial, there are M complex valued roots and each one constitutes a mode of the solution. As a summary, any root of the characteristic equation paired with an arbitrary coefficient constitutes a possible mode for the non-trivial solution, $y_h[n] = Ar^n$.

If we go back the problem of Fibonacci numbers, we have the difference equation of $y[n] = y[n-1] + y[n-2]$. The difference equation does not have any input; hence it is already a homogeneous difference equation. By substituting $y[n] = Ar^n$ into the difference equation, we can get the characteristic equation as $r^2 - r^1 - 1 = 0$. The roots of this equation are $r_1 = \frac{1-\sqrt{5}}{2}$ and $r_2 = \frac{1+\sqrt{5}}{2}$. Hence, the general form of the homogenous solution is $y_h[n] = A_1 r_1^n + A_2 r_2^n$ for all n. Since the difference equation to be solved is already a homogenous difference equation (i.e. there is no particular solution, or the particular solution is 0 sequence). Then, $y[n] = A_1 r_1^n + A_2 r_2^n$ becomes the solution of the following difference equation:

$$\begin{aligned} &y[n] = y[n-1] + y[n-2] \\ &y[0] = 1 \\ &y[1] = 1 \end{aligned} \qquad . \qquad (3)$$

It should be noted that in the solution of $y[n] = A_1 r_1^n + A_2 r_2^n$; there are two undetermined coefficients (A_1, A_2). By properly selecting these coefficients, we can satisfy the initial conditions:

$$\begin{aligned} y[0] &= A_1 + A_2 = 1 \\ y[1] &= A_1 r_1 + A_2 r_2 = 1 \end{aligned}$$

If we do the calculations, we can get the solution for A_1, A_2 as follows:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{r_2 - r_1} \begin{bmatrix} r_2 & -1 \\ -r_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{r_2 - r_1} \begin{bmatrix} r_2 - 1 \\ 1 - r_1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -r_1 \\ r_2 \end{bmatrix}.$$

By substituting A_1, A_2 into $y[n] = A_1 r_1^n + A_2 r_2^n$, we can get the final solution as

$$y[n] = \frac{1}{\sqrt{5}}\left(-r_1^{n+1} + r_2^{n+1}\right) = \frac{1}{\sqrt{5}}\left\{-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right\}, \; n \geq 0. \qquad (4)$$

Let's check whether this solution correct or not. To do that we can use Matlab. From Matlab command prompt, let's define the function myfunc(n). Myfunc has the only one argument (which is n) and numerically evaluates $y[n]$ given in (4):

```
>> myfunc = @(n) 1/sqrt(5)*(-((1-sqrt(5))/2)^(n+1) + ((1+sqrt(5))/2)^(n+1));
>> [myfunc(0), myfunc(1), myfunc(2), myfunc(3), myfunc(4), myfunc(5)]
ans =
    1.0000    1.0000    2.0000    3.0000    5.0000    8.0000
```

It seems that we have the right solution. ☺

3. Particular Solution

The particular solution of a difference equation is the part of the solution due to the applied input. For the CCDE difference equation studied here, the particular solution for a specific class of inputs is examined.

$$y[n]+\sum_{k=1}^{M}\alpha_k y[n-k]=f[n]. \qquad (5)$$

When the input $f[n]$ in (5) is of exponential type; that is $f[n]=Ar^n n^K$ (K is a non-negative integer, r is a complex number) the particular solution has the same type. It should be noted that this class is fairly general and includes the inputs such as polynomial type input (when $r=1$) such as unit step, ramp, etc. and exponential functions (when $K=0$) also cosines and sines functions (when r is purely imaginary).

The particular solution for this class of inputs can be easily found by guessing the form of the particular solution (which is the exponential family) and finding the undetermined coefficients. We present some examples to clarify the discussion:

Fibonacci Sequence Generator with External Input: We use the difference equation generating the Fibonacci numbers; but include the input $f[n]$ on the right hand side of the difference equation, that is

$$y[n]=y[n-1]+y[n-2]+f[n].$$

i. $f[n]=\frac{1}{3}2^n$

We make the following guess for the particular solution $y_p[n]=A2^n$. When $y_p[n]$ is substituted into the difference equation, we get $A2^n=A2^{n-1}+A2^{n-2}+\frac{1}{3}2^n$. From this equation, we can solve for A and get $A=\frac{4}{3}$. The particular solution becomes $y_p[n]=\frac{4}{3}2^n$.

ii. $f[n]=\gamma^n$

We make the following guess for the particular solution $y_p[n]=A\gamma^n$. When $y_p[n]$ is substituted into difference equation, we get $A\gamma^n=A\gamma^{n-1}+A\gamma^{n-2}+\gamma^n$. From this equation, we can solve for A and get $A=\frac{\gamma^2}{\gamma^2-\gamma^1-1}$. The particular solution becomes $y_p[n]=\frac{\gamma^2}{\gamma^2-\gamma^1-1}\gamma^n$. Note that the result for part i can be derived from this general result by setting $\gamma=2$.

iii. $f[n]=\left(\frac{1+\sqrt{5}}{2}\right)^n$

We make the following guess for the particular solution $y_p[n]=A\left(\frac{1+\sqrt{5}}{2}\right)^n$.

We may think that it is a good idea to use the general result part ii and write $A=\frac{\gamma^2}{\gamma^2-\gamma^1-1}$ for $\gamma=\frac{1+\sqrt{5}}{2}$; but the value for γ is substituted in $A=\frac{\gamma^2}{\gamma^2-\gamma^1-1}$; the denominator of A becomes zero! Hence, we can not find a value for A and therefore, $y_p[n]=A\left(\frac{1+\sqrt{5}}{2}\right)^n$ is not the right form for the particular solution.

Careful readers should have recognize that the guess of $A\left(\frac{1+\sqrt{5}}{2}\right)^n$ coincides with the homogenous solution for Fibonacci number generation system. Hence, with this guess left hand side of the difference equation becomes zero (since it is the homogenous solution!). Therefore, $y_p[n]=A\left(\frac{1+\sqrt{5}}{2}\right)^n$ can not be correct guess for the particular solution.

As a summary, We recognize that $y_p[n]=Ar_2^{\,n}$, where $r_2=\frac{1+\sqrt{5}}{2}$ is one of the roots of characteristic polynomial can not be the right form for the particular solution. Following the lead from our differential equation knowledge, we suggest to try $y_p[n]=Ar_2^{\,n}n$ as the particular solution.

When $y_p[n]=Ar_2^{\,n}n$ is substituted in the difference equation, we get

$$Ar_2^{\,n}n=Ar_2^{\,n-1}(n-1)+Ar_2^{\,n-2}(n-2)+r_2^{\,n}.$$

Dividing both sides by $(r_2)^{n-2}$, we get the following:

$$Ar_2^{\,2}n=Ar_2^{\,1}(n-1)+A(n-2)+r_2^{\,2}$$

$$An(r_2^{\,2}-r_2^{\,1}-1)=-Ar_2^{\,1}-2A+r_2^{\,2}$$

Since r_2 satisfies characteristic equation, that is $r_2^{\,2}-r_2^{\,1}-1=0$, the left hand side of the equation above is 0. This gives the value of A as $A=\frac{(r_2)^2}{r_2+2}$ and we can write the particular solution as $y_p[n]=\frac{n}{r_2+2}r_2^{\,n+2}$.

iv. $f[n] = n^2 + \frac{n}{2}$

We make the following guess for the particular solution $y_p[n] = An^2 + Bn + C$. When $y_p[n]$ is substituted into difference equation, we get

$$An^2 + Bn + C = A(n-1)^2 + B(n-1) + C + A(n-2)^2 + B(n-2) + C + n^2 + \frac{n}{2}.$$

By expanding the squares and collecting like-powers together we get:

$$(-C + 3B - 5A) + (6A - B)n - An^2 = n^2 + \frac{n}{2}$$

Since the equation above should be satisfied for all n values, the solution is achieved only when the coefficients of the like-powers on each side are identical:

$$\begin{aligned} (-C + 3B - 5A) &= 0 \\ (6A - B) &= \frac{1}{2}. \\ -A &= \frac{1}{2} \end{aligned} \qquad (6)$$

The solution of the equation system is $A = -\frac{1}{2}, B = -\frac{7}{2}, C = 8$ and the particular solution becomes $y_p[n] = -\frac{1}{2}n^2 - \frac{7}{2}n + 8$.

v. $f[n] = 2^n + n^2 + \frac{n}{2}$

We can make the following guess for the particular solution $y_p[n] = An^2 + Bn + C + D2^n$ and try to find the coefficients as we did before. Instead of doing this, we may recognize that $f[n] = 2^n + n^2 + \frac{n}{2}$ is nothing but the three times of $f[n]$ given in part i plus f[n] given in part iv When we repeat the calculations, we get 4 equations with 4 unknowns from which we can solve for unknown; but, since the function $D2^n$ and polynomials $An^2 + Bn + C$ does not "mix" with each other; we should get the equation set given in (6) for the solution of A. Hence, $A = -\frac{1}{2}, B = -\frac{7}{2}, C = 8$. Similarly, the unknown D should be three times of the coefficient determined in part i. (Why)

This discussion shows (and hopefully convinces you) that the particular solution of a difference equation obeys linearity. Hence, if several inputs are superposed at input-side of the system; the particular solution can be formed by superposition of the individual particular solution to each input.

4. Examples

Example 1: Verify the formula $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$.

Let's call the sum up to n as $s[n] = \sum_{k=0}^{n} k$. With this definition, it is easy to see that $s[n] = s[n-1] + n$. Since the difference equation is first order, we need only one initial condition to uniquely specify the solution. We may set the initial condition as $s[0] = 0$ and the problem reduces the solution of the following difference equation:

$$s[n] = s[n-1] + n$$
$$s[0] = 0$$

We start with the homogenous solution: $s_h[n] = s_h[n-1]$. We make the guess of $s_h[n] = Ar^n$ and get the characteristic equation as $r - 1 = 0$. Hence, the homogenous solution is $s_h[n] = A$.

For the particular solution $s_p[n] = s_p[n-1] + n$, we make the guess of $s_p[n] = Bn^2 + Cn$ (Why?). Once the guess is substituted into difference equation, we get

$$Bn^2 + Cn = B(n-1)^2 + C(n-1) + n.$$

Expanding the brackets, we get $2Bn + C - B = n$; from which, we get the particular solution as $B = C = \frac{1}{2}$.

The complete solution of the difference equation is then

$$s_{comp}[n] = s_h[n] + s_p[n] = A + \frac{1}{2}(n^2 + n).$$

At the final step, we need to set the undetermined coefficient (A) of the homogenous solution. To do that, we use initial condition $s[0] = 0$ and get $A = 0$. The complete solution becomes $s[n] = \frac{1}{2}(n^2 + n) = \frac{n(n+1)}{2}$.

Example 2: Verify the formula $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. (This is left to readers.)

Example 3: Find the complete solution of $y[n] = \frac{1}{4} y[n-1] + n$ for $n \geq 0$ with the initial condition of $y[-1] = 10$.

Particular Solution: Let $y_p[n] = An + B$, then upon the substitution of the guess into the difference equation, we get $An + B = \frac{1}{4} A\big((n-1) + B\big) + n$ for $n > 0$. From the last equation, we can solve for A and B as $A=\frac{4}{3}$ and $B=-\frac{4}{9}$. The particular solution becomes $y_p[n]=\frac{4}{3}n-\frac{4}{9}$.

Homogenous Solution: The solution of $y[n]=\frac{1}{4}y[n-1]$ is $y_h[n]=A\left(\frac{1}{4}\right)^n$.

Complete Solution: The complete solution is $y_h[n]+y_p[n]=A\left(\frac{1}{4}\right)^n+\frac{4}{3}n-\frac{4}{9}$. At this point, the reader is advised to substitute the complete solution into the difference equation and to verify that this solution indeed satisfies the recursion for any value for A. To set the value for A, we need to use the initial condition. We can use $y[-1]=10$ (or run the recursion one step from the initial condition and get $y[0]=10/4$) or y[0] or any other subsequent sample to find the value for A. Using $y[0]=10/4$, we get $y[0]=\frac{10}{4}=A-\frac{4}{9}$ and find $A=\frac{53}{18}$. The solution becomes $y[n]=\frac{53}{18}\left(\frac{1}{4}\right)^n+\frac{4}{3}n-\frac{4}{9}$, for $n\geq 0$.

<u>**Example 3:**</u> Kivanc prefers to have a saving account for the investment of his monthly income. The monthly return on the saving account is α units, that is the bank gives α -1 TL interest at the end of month for every 1 TL invested in the account. Hence, at the end of the one month 1 TL becomes α TL. On the account saturation day (or the interest collection day), Kivanc may withdraw or deposit cash into this account and the amount remains after withdrawl/deposit operation is invested for another month.

If Kivanc's initial deposit in the bank account is I_{-1} TL and his account deposit/withdraws at the end of n'th month is shown with $x[n]$, express Kivanc's total sum in the bank account at the n'th month?

Solution: If we denote the total sum at the end of the n'th month as $y[n]$, then on the day of interest payment, the total sum in the bank account becomes:

$$y[n]=\alpha y[n-1]+x[n]$$

Here, the term of $\alpha y[n-1]$ is due to invested income and $x[n]$ is the deposit/withdraw to the account on the account saturation day. We also know that the initial sum in the bank account is I_{-1}. This information sets the initial condition for the difference equation as $y[-1]=I_{-1}$. Kivanc's problem is then the solution of the following difference equation:

$$\begin{aligned} y[n]&=\alpha y[n-1]+x[n], \\ y[-1]&=I_{-1} \end{aligned} \quad . \qquad (7)$$

Since x[n] is not provided in an analytical form, we can not use guess-substitute-verify method to find the particular solution; but running the recursion given in (7) a few times gives us some hint about the form of the solution:

$$
\begin{aligned}
y[-1] &= I_{-1} \\
y[0] &= \alpha I_{-1} + x[0] \\
y[1] &= \alpha^2 I_{-1} + \alpha x[0] + x[1] \\
y[2] &= \alpha^3 I_{-1} + \alpha^2 x[0] + \alpha x[1] + x[2] \cdot \\
&\dots \\
y[n] &= \alpha^n I_{-1} + \sum_{k=0}^{n} x[k]\alpha^{n-k}, n \geq 0
\end{aligned}
\tag{8}
$$

The solution has a natural interpretation. The first term in the result of $y[n] = \alpha^n I_{-1} + \sum_{k=0}^{n} x[k]\alpha^{n-k}, n \geq 0$, that is $\alpha^n I_{-1}$, corresponds the contribution of the initial investment on the total sum after n months. The initial sum is compounded n times. Each term of the summation of $\sum_{k=0}^{n} x[k]\alpha^{n-k}$, such as $x[k]\alpha^{n-k}$, is the component of related with the k'th month deposit $x[k]$ (or withdraw if it is a negative value) at the n'th month.

It should be clear that $\alpha^n I_{-1}$ corresponds to the homegenous solution and $\sum_{k=0}^{n} x[k]\alpha^{n-k}$ corresponds to the particular solution. Indeed, the particular and homegenous solutions for this problem given in (8) is rather special. At this point, we would like to note that the particular solution of a difference equation is not unique; since by adding a multiple of a homogenous solution to a particular solution; we can get another particular solution. Therefore, it is proper to denote the particular solutions with the article of "a" and state that "a particular solution to this problem is …" instead of "the particular solution …"

Returning back to the problem, we can note that the given particular solution in (8) , that is $\sum_{k=0}^{n} x[k]\alpha^{n-k}$ for $n \geq 0$, is the solution when the initial deposit is equal to zero. (To see this, just insert $I_{-1} = 0$ in $y[n] = \alpha^n I_{-1} + \sum_{k=0}^{n} x[k]\alpha^{n-k}, n \geq 0$). This particular solution is called zero-state solution. (This should ring some bells for the students who are not atonal to the circuit theory courses.)

Similarly, in the absence of input, that is $x[n] = 0$ for $n \geq 0$; the solution is only due to initial deposit I_{-1}; hence the special homogenous solution of $y[n] = \alpha^n I_{-1}$ is called the zero-input solution.

Let's try to drive these results using the approach we have discussed in these notes. The main idea is to decompose the input in terms of impulses and use the impulse response of the system to find the zero-state solution. (One more time, the zero-state solution is a special particular solution.)

Impulse response calculation: It should be remembered that impulse response is calculated when the system is initially at rest.

$$h[n] = \alpha h[n-1] + \delta[n]$$
$$h[-1] = 0 \quad . \qquad (9)$$

For the calculation of the solution, we make the following guess $h[n] = Ar^n u[n]$ and substitute the guess into the difference equation and get $Ar^n u[n] = \alpha A r^{n-1} u[n-1] + \delta[n]$.

It should be remembered the recursion in (9) should be valid for all $n \geq 0$. When we set $n = 0$ in $Ar^n u[n] = \alpha A r^{n-1} u[n-1] + \delta[n]$, we get $A = 1$. For $n \geq 1$, the equation of $Ar^n u[n] = \alpha A r^{n-1} u[n-1] + \delta[n]$ reduces to $r^n = \alpha r^{n-1}$ from which we can get $r = \alpha$. Hence, we have found that the choice of $A = 1$ and $r = \alpha$ in $h[n] = Ar^n u[n]$, results in the impulse response, $h[n] = \alpha^n u[n]$.

Expressing the input x[n] in terms of impulses: The input x[n] can be written in terms of impulses as $x[n] = \sum_{k=0}^{\infty} x[k]\delta[n-k]$. Once this is done, the difference equation in (7) can be written as follows:

$$y[n] = \alpha y[n-1] + \sum_{k=0}^{\infty} x[k]\delta[n-k]$$
$$y[-1] = I_0 \quad . \qquad (10)$$

Finding a particular solution: A particular solution of (10) can be written from the impulse response. When it is written, we get $y_p[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$. (Please read part v. of particular solution example titled "Fibonacci Sequence Generator With External Input", if you are having difficulty at this step.) So the particular solution is $y_p[n] = \sum_{k=0}^{\infty} x[k]h[n-k] = \sum_{k=0}^{\infty} x[k]\alpha^{n-k}u[n-k] = \sum_{k=0}^{n} x[k]\alpha^{n-k}$.

Finding the homogenous solution: The homogenous solution for $y[n] = \alpha y[n-1] + x[n]$ is $y_h[n] = A\alpha^n$.

Finding the complete solution: The complete solution becomes

$$y[n] = y_h[n] + y_p[n] = A\alpha^n + \sum_{k=0}^{n} x[k]\alpha^{n-k} .$$

We are almost there! Since $y[-1] = I_{-1}$, we have $A = \alpha I_{-1}$ (or you may use $y[0] = \alpha I_{-1} + x[0]$ and get the same value for A). Substituting the value for A in the complete solution, we get $y[n] = I_{-1}\alpha^{n+1} + \sum_{k=0}^{n} x[k]\alpha^{n-k}$ for $n \geq 0$, which is the result that we are trying to show.

<u>**Example 4:**</u> Beren (a friend of Kivanc) wants to calculate the digit sum of the non-negative integers with n decimal digits. As an example, for $n = 1$, the set of integers with a single decimal digit are $\{0,1,2,3,4,5,6,7,8,9\}$ and their sum is 45. For $n = 2$, the set becomes $\{00,01,02,03,04,05,06,07,08,09,10,11,...,99\}$; the sum of their digits is

$450+10x0+10x1+10x2+...10x9=900$. (why?) The question is to find the expression for the digit sum of n digit non-negative integers.

Solution: Let the digit sum for n digit non-negative integers be $s[n]$. We know that $s[1]=45$ and $s[2]=900$. We need the find a relation between the elements of the sequence $s[n]$. The relation can be given as (after some thought) as follows:

$$s[n]=10s[n-1]+10^{n-1}45$$

To check the relation, we can calculate $s[2]=10s[1]+450$ using $s[1]=45$. This gives the value of $s[2]=900$, matching Beren's calculation in the problem definition. If the recursion is repeated for the next digit, we get $s[3]=10s[2]+4500=13500$. To verify this result, we can use the following Matlab line:

```
>> vec=0:999; vec2 = num2str(vec); dum = abs(vec2); dum(dum==32)=[]; dum= dum-48;
>> sum(dum)
ans =
    13500
```

(The Matlab line can be a little cryptic at first sight. To better understand the code, please experiment with it.) It seems that the recursion gives the correct results.

Hence, Beren's problem reduces to the solution of the following difference equation:

$$s[n]=10s[n-1]+10^{n-1}45, \qquad n\geq 2$$
$$s[1]=45$$

Approach 1: The homogenous solution is $s_h[n]=A10^n$. A particular solution (not the particular solution) is $s_p[n]=Bn10^n$. By substituting $s_p[n]$ into the difference equation, we can get $B=4.5$. Then the complete solution is $s[n]=A10^n+4.5n10^n$. To satisfy the condition of $s[1]=45$, we need to set $A=0$. The final solution is then $s[n]=4.5n10^n$ for $n\geq 1$.

Approach 2: Let's use the results of Example 3 (Kivanc's problem) for the solution. To do that we call the sequence $10^{n-1}45$ for $n\geq 2$ as the input $x[n]$. The complete solution is then

$$s[n]=A\alpha^n+\sum_{k=1}^{n}x[k]\alpha^{n-k}, n\geq 1$$

and we have $\alpha=10$.

Let's calculate the zero-state part of the solution, i.e. $\sum_{k=1}^{n}x[k]\alpha^{n-k}$. Since, $\alpha=10$; and $x[n]=4510^{n-1}=4.5x10^n$, $\sum_{k=1}^{n}x[k]\alpha^{n-k}=4.5\sum_{k=1}^{n}10^k10^{n-k}=4.5x10^n\sum_{k=1}^{n}1=4.5(n-1)10^n$ for $n\geq 1$.

Then the solution becomes $s[n]=A10^n+4.5(n-1)10^n, n\geq 1$. Yet, there is an undetermined coefficient (A) to determine. Using $s[1]=45$ together with

$s[1] = A10^1 + 4.5(1-1)10^1 = A$, we can get the unknown as $A = 4.5$. The final solution becomes $s[n] = 4.510^n + 4.5(n-1)10^n, n \geq 1$ which can be simplified to $s[n] = 4.5n10^n, n \geq 1$.

Example 5: Tugba (a friend of Kıvanc) has a determinant problem. Her problem is to calculate the determinant of the following $N \times N$ Toeplitz tri-diagonal matrix $\boldsymbol{M}$:

$$\boldsymbol{M}_{N\times N} = \begin{bmatrix} a & b & 0 & \dots & 0 & 0 & 0 \\ c & a & b & \dots & 0 & 0 & 0 \\ 0 & c & a & b & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & c & a & b & 0 \\ 0 & 0 & \dots & 0 & c & a & b \\ 0 & 0 & \dots & 0 & 0 & c & a \end{bmatrix}_{N\times N} .$$

Tugba expands the determinant into its co-factors along the first row,

$$|\boldsymbol{M}_{\boldsymbol{N}\times\boldsymbol{N}}| = a \begin{vmatrix} a & b & \dots & 0 & 0 & 0 \\ c & a & b & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & c & a & b & 0 \\ 0 & \dots & 0 & c & a & b \\ 0 & \dots & 0 & 0 & c & a \end{vmatrix} - b \begin{vmatrix} c & b & \dots & 0 & 0 & 0 \\ 0 & a & b & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & c & a & b & 0 \\ 0 & \dots & 0 & c & a & b \\ 0 & \dots & 0 & 0 & c & a \end{vmatrix}.$$

and denotes $|\boldsymbol{M}_{N\times N}|$ with D_N, i.e. $D_N = |\boldsymbol{M}_{N\times N}|$. Then observes that, the expression above can be written as

$$D_N = aD_{N-1} - b \begin{vmatrix} c & b & \dots & 0 & 0 & 0 \\ 0 & a & b & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & c & a & b & 0 \\ 0 & \dots & 0 & c & a & b \\ 0 & \dots & 0 & 0 & c & a \end{vmatrix}.$$

To calculate the minor, the unknown determinant given above; Tugba expands the determinant along the first row, one more time:

$$\begin{vmatrix} c & b & \dots & 0 & 0 & 0 \\ 0 & a & b & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & c & a & b & 0 \\ 0 & \dots & 0 & c & a & b \\ 0 & \dots & 0 & 0 & c & a \end{vmatrix} = c \begin{vmatrix} & & & & \\ a & b & \vdots & \vdots & \vdots \\ \ddots & \ddots & \ddots & \vdots & \vdots \\ \dots & c & a & b & 0 \\ \dots & 0 & c & a & b \\ \dots & 0 & 0 & c & a \end{vmatrix} - b \begin{vmatrix} 0 & b & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & c & a & b & 0 \\ 0 & 0 & c & a & b \\ 0 & 0 & 0 & c & a \end{vmatrix}$$

Finally, this calculation results in

$$\begin{vmatrix} c & b & \dots & 0 & 0 & 0 \\ 0 & a & b & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & c & a & b & 0 \\ 0 & \dots & 0 & c & a & b \\ 0 & \dots & 0 & 0 & c & a \end{vmatrix} = cD_{N-2}.$$

Combining all the results, she gets the following recursion

$$D_N = aD_{N-1} - bcD_{N-2}$$

with the initial conditions of $D_1 = a$ and $D_2 = a^2 - bc$ for the determinant expression.

As an example, let's take $a = 2,\ b = 1$ and $c = -3$. With these parameters, the recursion becomes

$$D_N = 2D_{N-1} + 3D_{N-2}, \qquad N \geq 3$$
$$D_1 = 2,$$
$$D_2 = 7.$$

The solution is then $D_N = \frac{1}{4}(3^{N+1} + (-1)^N)$. Good for Tugba!